MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

MAA American Mathematics Competitions

31st Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 17, 2015

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions should be addressed to:

Prof. Norbert Kuenzi, AMC 8 Chair
934 Nicolet Ave
Oshkosh, WI 54901-1634

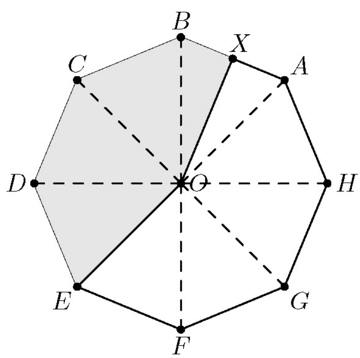
Orders for prior year exam questions and solutions pamphlets should be addressed to:
MAA American Mathematics Competitions
Attn: Publications
PO Box 471
Annapolis Junction, MD 20701

1. **Answer (A):** The floor is $\frac{12}{3} = 4$ yards long and $\frac{9}{3} = 3$ yards wide, so it will take $4 \times 3 = 12$ square yards of carpet to cover it.

OR

The area of the floor is 12×9 square feet. There are $3^2 = 9$ square feet in a square yard, so the number of square yards required is $\frac{12\times 9}{9} = 12$.

2. **Answer (D):** The octagon can be divided into 8 congruent triangles, 3 of which are $\triangle BOC$, $\triangle COD$, and $\triangle DOE$. The area of $\triangle XOB$ is half the area of one of these, so the fraction of the area of the octagon that is shaded is $3 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} = \frac{7}{16}$.



3. **Answer (D):** Jill takes $\frac{1}{10}$ of an hour, or 6 minutes, to get to the pool, and Jack takes $\frac{1}{4}$ of an hour, or 15 minutes, so Jill arrives $15 - 6 = 9$ minutes before Jack.

4. **Answer (E):** There are 2 ways to seat the boys, one on each end, and $3 \cdot 2 \cdot 1 = 6$ ways to seat the three girls in the middle. So there are $2 \cdot 6 = 12$ possible arrangements.

5. **Answer (A):** The range is the high score minus the low score, so the range changes from 31 to 33. The range is the only listed statistic that will increase. Because 40 is the lowest score for the season, it will cause the mean to decrease. The median value of the first 11 games is the 6th highest score, or 58. The median value of the first 12 games will be the average of the 6th highest and 7th highest scores, or $(58 + 58)/2 = 58$, so no change will occur in the median. Similarly, the score that occurs most frequently in either situation is 58, so the mode will not change. The mid-range is the average of the highest score and the lowest score. The mid-range of the first 11 games is $(73 + 42)/2 = 57.5$. The mid-range of the first 12 games is 56.5, a decrease from 57.5.

6. **Answer (B):** Let D be the midpoint of side $\overline{AC}$. Then $\overline{BD}$ is the altitude to $\overline{AC}$ and $\triangle BDC$ is a right triangle with $BC = 29$ and $DC = 21$. So $BD = \sqrt{29^2 - 21^2} = \sqrt{400} = 20$. The area of $\triangle ABC = \frac{1}{2} \cdot 20 \cdot 42 = 420$.

OR

Heron's formula gives the area of a triangle in terms of the lengths of its sides. If the side lengths are a, b and c, then let $s = \frac{a+b+c}{2}$. The area is then $\sqrt{s(s-a)(s-b)(s-c)}$. In this problem, $s = \frac{29+29+42}{2} = 50$, and the area is $\sqrt{50 \cdot 21 \cdot 21 \cdot 8} = 21\sqrt{400} = 21 \cdot 20 = 420$.

7. **Answer (E):** The nine possible equally likely outcomes are:

$$(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$$

In five of the nine outcomes the product is even. Therefore the probability is $\frac{5}{9}$.

OR

The only way the product of the two values could be odd is if an odd number is drawn from each box. The probability that this occurs is $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$. So the probability that the product is even is $1 - \frac{4}{9} = \frac{5}{9}$.

8. **Answer (D):** Let t be the length of the third side of the triangle. By the Triangle Inequality, $t < 5 + 19 = 24$. So the perimeter $5 + 19 + t < 5 + 19 + (5 + 19) = 48$.

9. **Answer (D):** Note that Janabel has sold a total of 1 widget after 1 day, $1 + 3 = 4 = 2^2$ after 2 days and $1 + 3 + 5 = 3^2$ widgets after 3 days. It can be shown that this pattern continues so that after 20 days, Janabel has sold a total of $20^2 = 400$.

OR

The sum of the first 20 odd numbers $1, 3, 5, \ldots, 39$ is needed. The sum of the first and last is 40, the sum of the second and 19th is also 40, and in fact, there are 10 pairs of numbers that each add up to 40. Thus the required sum is 400.

10. **Answer (B):** The thousands position can be filled by the digits 1 through 9 (0 is excluded). Without repetition, the hundreds position can be filled with any of the remaining 9 digits (including 0). Similarly without repetition, the tens position and ones position can be filled with the remaining 8 and 7 digits, respectively. Thus there are $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ integers between 1000 and 9999 that have distinct digits.

11. **Answer (B):** The first symbol can be any of the 5 vowels, the second can be any of the 21 consonants, the third can be any of the 20 other consonants, and the fourth can be any of the 10 digits. The total number of possible license plates is $5 \cdot 21 \cdot 20 \cdot 10 = 21{,}000$. Only one plate will read "AMC8", so the probability is $\frac{1}{21{,}000}$.

12. **Answer (C):** Each of the 12 edges is parallel to 3 other edges giving 36 possible pairs of parallel edges. But each pair of parallel edges is counted twice in this process, so there are 18 pairs of parallel edges.

OR

There are 6 pairs of parallel edges related to $\overline{AB}$ ($\overline{AB} \| \overline{EF}$, $\overline{AB} \| \overline{HG}$,$\overline{AB} \| \overline{DC}$, $\overline{EF} \| \overline{HG}$, $\overline{EF} \| \overline{DC}$, $\overline{HG} \| \overline{DC}$). Similarly there are 6 pairs of parallel edges related to $\overline{AE}$ and 6 pairs of parallel edges related to $\overline{AD}$ for a total of 18 pairs of parallel edges.

13. **Answer (D):** If the average of the remaining 9 numbers is 6, then their sum is 54. Because the sum of the numbers in the original set is 66, the sum of the two numbers removed must be 12. There are five such subsets: $\{1, 11\}$, $\{2, 10\}$, $\{3, 9\}$, $\{4, 8\}$, and $\{5, 7\}$.

14. **Answer (D):** The sum of 4 consecutive odd integers is always a multiple of 8, $(2n-3)+(2n-1)+(2n+1)+(2n+3) = 8n$. Among the given choices, only 100 is not a multiple of 8. The other four numbers can each be written as the sum of four consecutive odd numbers:

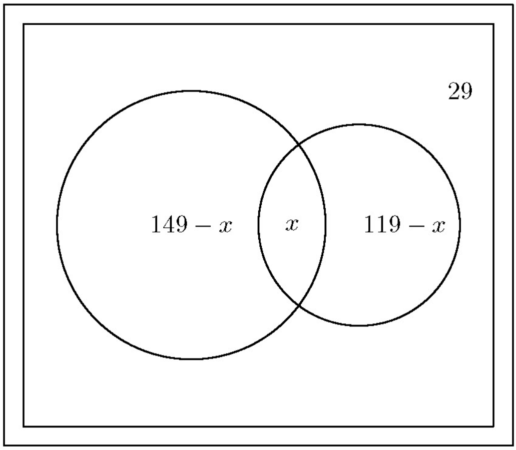
$$\begin{aligned} 16 &= 1+3+5+7 \\ 40 &= 7+9+11+13 \\ 72 &= 15+17+19+21 \\ 200 &= 47+49+51+53 \end{aligned}$$

15. **Answer (D):** The sum $149 + 119 + 29 = 297$ counts the number of students who voted for both issues twice. So the number who voted in favor of both issues is $297 - 198$ or 99.

OR

In the diagram below, the left circle represents the 149 students who voted for the first issue, and the right circle represents the 119 students who voted for the second issue. Let x be the number of students who voted for both issues. Then $149 - x$ students voted for the first issue but not the second, $119 - x$ students voted for the second issue but not the first and 29 students voted against both issues. The sum of the numbers in the diagram must be 198, so

$$\begin{aligned}(149 - x) + x + (119 - x) + 29 &= 198,\\ 297 - x &= 198,\\ x &= 99.\end{aligned}$$



16. **Answer (B):** 2 out of every 5 sixth graders are paired with a ninth grade buddy, and 2 out of every 6 ninth graders are paired with a sixth grade buddy. (The ratios are now expressed so that the number of sixth graders matches the number of ninth graders.) So 4 out of every 11 students are in the mentoring program. The fraction is $\frac{4}{11}$.

OR

Suppose that n sixth graders are paired with n ninth graders. Then the total number of sixth graders is $\frac{5}{2}n$, the total number of ninth graders is $3n$, and the total number of sixth and ninth graders is $\frac{5}{2}n+3n = \frac{11}{2}n$. There are $2n$ students in the mentoring program, which is $\dfrac{2n}{\frac{11}{2}n} = \dfrac{4}{11}$ of the total number of students.

17. **Answer (D):** Because the new time is $\frac{12}{20} = \frac{3}{5}$ of the original time, the new speed must be $\frac{5}{3} = 1\frac{2}{3}$ of the original speed. Then the additional 18 miles per hour must be $\frac{2}{3}$ of the original speed, which is then 27 mph. In 20 minutes, Jeremy's father travels $\frac{1}{3} \cdot 27 = 9$ miles.

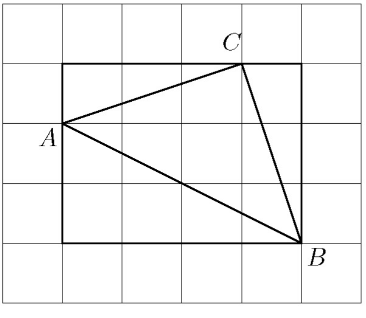
OR

Let r be Jeremy's original speed in miles per hour. Twenty minutes is $\frac{20}{60} = \frac{1}{3}$ of an hour and twelve minutes is $\frac{12}{60} = \frac{1}{5}$ of an hour. Then $\frac{1}{3}r = \frac{1}{5}(r + 18)$, so $5r = 3r + 54$, and $r = 27$. Thus the distance to the school is $\frac{1}{3}r = 9$ miles.

18. **Answer (B):** The middle number in an arithmetic sequence with 5 terms is the average of the first and last numbers. The average of 1 and 25 is 13. The average of 17 and 81 is 49. Thus, X is the average of 13 and 49, or 31. Alternatively, $X = \frac{9+53}{2} = 31$. In fact X is the average of the four corner entries.

1		13		25
9		31		53
17		49		81

19. **Answer (A):** The triangle is inscribed in a 4×3 rectangle with vertices at $(1, 1)$, $(1, 4)$, $(5, 4)$, and $(5, 1)$. Three triangular regions are inside the 4×3 rectangle but outside $\triangle ABC$. The area of the lower-left triangle is $\frac{1}{2} \cdot 4 \cdot 2 = 4$ square units. The area of the upper-left triangle is $\frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$ square units. The area of the third triangle is also $\frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$ square units. So the area of $\triangle ABC$ is $12 - 4 - \frac{3}{2} - \frac{3}{2} = 5$ square units. The area of the 6×5 grid is 30 square units. Thus, the fraction covered by the triangle is $\frac{5}{30} = \frac{1}{6}$.



OR

Pick's Theorem says that the area of a polygonal region whose vertices are at lattice points (points whose coordinates are integers) is given by $A = I + \frac{1}{2}B - 1$ where I is the number of lattice points in the interior of the region and B is the number of lattice points on the boundary. Referring to the figure above, there are $B = 4$ lattice points on the boundary of $\triangle ABC$ at $(1,3)$, $(3,2)$, $(5,1)$, and $(4,4)$. There are $I = 4$ points with integer coordinates in the interior of $\triangle ABC$ at $(2,3)$, $(3,3)$, $(4,2)$, and $(4,3)$. Then the area of $\triangle ABC$ is $I + \frac{1}{2}B - 1 = 4 + 2 - 1 = 5$ square units. As before, the fraction of the rectangle covered by the triangle is $\frac{5}{30} = \frac{1}{6}$.

20. **Answer (D):** If Ralph buys 6 pairs of \$1 socks, then the other 6 pairs of socks would cost at least \$19 making the total cost more than \$24. Buying fewer than 6 pairs of \$1 socks would make Ralph's cost even higher. If he bought 8 pairs of \$1 socks, then the other 4 pairs would cost less than \$16 making the total cost less than \$24. Buying more than 8 pairs of \$1 socks would make his total cost even lower. So Ralph bought 7 pairs of \$1 socks, 3 pairs of \$3 socks, and 2 pairs of \$4 socks.

OR

Let a, b and c be the number of pairs of \$1, \$3 and \$4 socks, respectively. Then $a + b + c = 12$ and $a + 3b + 4c = 24$. Subtracting the first equation from the second gives $2b + 3c = 12$. Since 3 is a factor of both 12 and $3c$, 3 must also be a factor of $2b$. Since $c > 0$, it follows that $b = 3$, $c = 2$, and $a = 7$.

21. **Answer (C):** The area of the square $ABJI$ is 18 and $\triangle KJB$ is equilateral, so $KB = JB = \sqrt{18} = 3\sqrt{2}$. The area of the square $FEHG$ is 32, so $BC = FE = \sqrt{32} = 4\sqrt{2}$. Each interior angle of the hexagon is 120°, so $\angle KBC = 360^\circ - 60^\circ - 90^\circ - 120^\circ = 90^\circ$ and $\triangle KBC$ is a right triangle. Its area is $\frac{1}{2} \cdot 3\sqrt{2} \cdot 4\sqrt{2} = 12$.

22. **Answer (C):** The number of students must be a multiple of 6 and also a multiple of 15. So the number of students must be divisible by the least common multiple of 6 and 15 which is 30. The divisors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30, so there are only 8 divisors. The divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60. So 60 has 12 divisors and 60 is the smallest possible number of students.

23. **Answer (D):**

The sum of the numbers is 35. So the 5 consecutive numbers in the cups must be 5, 6, 7, 8, and 9. It is impossible to get a sum of 5 or 7 using the slip with

3.5. Cup B needs a sum of 6, but it already has a slip with 3 on it so the slip with a 3.5 can't go there. Cup E needs a sum of 9, but with a slip with 2 in it the slip with 3.5 can't go there. The only place the slip with 3.5 on it can go is Cup D. One possibility is:

Cup A. 2, 3

Cup B. 3, 3

Cup C. 2.5, 4.5

Cup D. 2, 2.5, 3.5

Cup E. 2, 3, 4

24. **Answer (B):** The number of games played by a team is $3N + 4M = 76$. Because $M > 4$ and $N > 2M$ it follows that $N > 8$. Because 4 divides both 76 and $4M$, 4 must divide $3N$ and hence N. If $N = 12$ then $M = 10$ and the condition $N > 2M$ is not satisfied. If $N \geq 20$ then $M \leq 4$ and the condition $M > 4$ is not satisfied. So the only possibility is $N = 16$ and $M = 7$. So each team plays $3 \cdot 16 = 48$ games within its division and $4 \cdot 7 = 28$ games against the other division.

OR

The total number of games played by each team is $3N + 4M = 76$. Make a chart of possibilities with $M > 4$:

M	$4M$	$76 - 4M = 3N$	N
5	20	56	(not an integer)
6	24	52	(not an integer)
7	28	48	16
8	32	44	(not an integer)
9	36	40	(not an integer)
10	40	36	12
11	44	32	(not an integer)
12	48	28	(not an integer)
13	52	24	8

Only $M = 7$ and $N = 16$ satisfy the conditions.

The case $M = 10$ and $N = 12$ violates the condition $N > 2M$.

So each team plays $3N = 48$ divisional games and $4M = 28$ games against the other division.

This is modeled on the Pioneer Baseball League with teams in Colorado, Idaho, Montana, and Utah.

25. **Answer (C):** Let $EQ = c$ and $TQ = s$ as indicated in the figure. Triangles QUV and FEQ are similar since $\angle FQE$ and $\angle QVU$ are congruent because

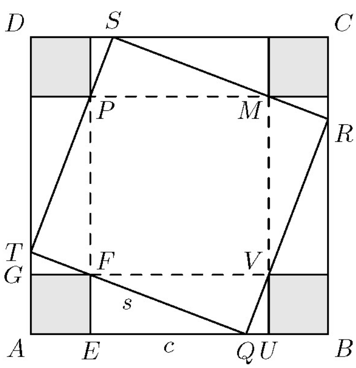
both are complementary to $\angle VQU$. So

$$\frac{QU}{UV} = \frac{FE}{EQ}$$

and thus $QU = \frac{1}{c}$. Then $AB = 1 + c + 1/c + 1 = 5$ and so $c + 1/c = 3$. Since the area of square $ABCD$ equals the sum of areas of square $QRST$, four unit squares, four $1 \times c$ triangles, and four $\frac{1}{c} \times 1$ triangles, it follows that

$$\begin{aligned} 25 &= s^2 + 4\left(1 + \frac{c}{2} + \frac{1}{2c}\right) \\ &= s^2 + 4 + 2\left(c + \frac{1}{c}\right) \\ &= s^2 + 4 + 2 \cdot 3 \end{aligned}$$

Therefore, the area of square $QRST = s^2 = 15$.



OR

Square $FVMP$ has area 9, the four triangles FQV, VRM, MSP, and PTF each have area $\frac{3}{2}$. So the area of square $STQR$ is $9 + 6 = 15$.

We thank the following donors for their generous support of the MAA American Mathematics Competitions, MOSP and the IMO

Patron's Circle

Akamai Foundation
Simons Foundation

Winner's Circle

American Mathematical Society
The D.E. Shaw Group
Dropbox
MathWorks
Two Sigma
Tudor Investment Corporation

Achiever's Circle

Art of Problem Solving
Jane Street Capital
Math for America

Sustainer's circle

Academy of Applied Science
Army Educational Outreach Program

Collaborator's Circle

American Statistical Association
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
Expii, Inc.
IDEA MATH
Mu Alpha Theta
National Council of Teachers of Mathematics
Society for Industrial and Applied Mathematics
Star League